# A NEW GENERATING FUNCTION OF ($q$-) BERNSTEIN TYPE POLYNOMIALS AND THEIR INTERPOLATION FUNCTION

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## Abstract

The main object of this paper is to construct a new generating function of the ($q$-) Bernstein type polynomials. We establish elementary properties of this function. By using this generating function, we derive recurrence relation and derivative of the ($q$-) Bernstein type polynomials. We also give relations between the ($q$-) Bernstein type polynomials, Hermite polynomials, Bernoulli polynomials of higher-order and the second kind Stirling numbers. By applying Mellin transformation to this generating function, we define interpolation of the ($q$-) Bernstein type polynomials. Moreover, we give some applications and questions on approximations of ($q$-) Bernstein type polynomials, moments of some distributions in Statistics.

CONTENTS

# 1. Introduction

In [4], Bernstein introduced the Bernstein polynomials. Since that time, many authors have studied on these polynomials and other related subjects (see cf. [1]-[21]), and see also the references cited in each of these earlier works. The Bernstein polynomials can also be defined in many different ways. Thus, recently, many applications of these polynomials have been looked for by many authors. These polynomials have been used not only for approximations of functions in various areas in Mathematics but also the other fields such as smoothing in statistics, numerical analysis and constructing Bezier curve which have many interesting applications in computer graphics (see cf. [4], [6], [13], [14], [15], [12], [16], [21]) and see also the references cited in each of these earlier works.

The ($q$-) Bernstein polynomials have been investigated and studied by many authors without *generating function*. So far, we have not found any generating function of ($q$-) Bernstein polynomials in the literature. Therefore, we will consider the following question:

*How can one construct **generating function** of ($q$-) Bernstein type polynomials?*

The aim of this paper is to give answer this question and to construct generating function of the ($q$-) Bernstein type polynomials which is given in Section 3. By using this generating function, we not only give recurrence relation and derivative of the ($q$-) Bernstein type polynomials but also find relations between higher-order Bernoulli polynomials, the second kind Stirling numbers and the Hermite polynomials. In Section 5, by applying Mellin transformation to the generating function of the ($q$-) Bernstein type polynomials, we define interpolation function, which interpolates the ($q$-) Bernstein type polynomials at negative integers.

## 2. Preliminary results related to the classical Bernstein, higher-order Bernoulli and Hermit polynomials, the second kind Stirling numbers

The Bernstein polynomials play a crucial role in approximation theory and the other branches of Mathematics and Physics. Thus in this section we give definition and some properties of these polynomials.

Let $f$ be a function on $[0, 1]$. The classical Bernstein polynomials of degree $n$ are defined by

$$\mathbb{B}_n f(x) = \sum_{j=0}^{n} f\left(\frac{j}{n}\right) B_{j,n}(x), \ 0 \leq x \leq 1, \tag{2.1}$$

where $\mathbb{B}_n f$ is called the Bernstein operator and

$$B_{j,n}(x) = \binom{n}{j} x^j (1-x)^{n-j}, \tag{2.2}$$

$j = 0, \ 1, \cdots, n$ are called the Bernstein basis polynomials (or the Bernstein polynomials of degree $n$). There are $n+1$ $n$th degree Bernstein polynomials. For mathematical convenience, we set $B_{j,n}(x) = 0$ if $j < 0$ or $j > n$ cf. ([4], [6], [8]).

If $f : [0, 1] \to \mathbb{C}$ is a continuous function, the sequence of Bernstein polynomials $\mathbb{B}_n f(x)$ converges uniformly to $f$ on $[0, 1]$ cf. [9].

A recursive definition of the $k$th $n$th Bernstein polynomials can be written as

$$B_{k,n}(x) = (1-x)B_{k,n-1}(x) + xB_{k-1,n-1}(x).$$

For proof of the above relation see [8].

For $0 \leq k \leq n$, derivative of the $n$th degree Bernstein polynomials are polynomials of degree $n-1$:

$$\frac{d}{dt}B_{k,n}(t) = n\left(B_{k-1,n-1}(t) - B_{k,n-1}(t)\right), \tag{2.3}$$

cf. ([4], [6], [8]). On the other hand, in Section 3, using our a new generating function, we give the other proof of (2.3).

Observe that the Bernstein polynomial of degree $n$, $\mathbb{B}_n f$, uses only the sampled values of $f$ at $t_{nj} = \frac{j}{n}$, $j = 0, 1, \cdots, n$. For $j = 0, 1, \cdots, n$,

$$\beta_{j,n}(x) \equiv (n+1)B_{j,n}(x), \ 0 \leq x \leq 1,$$

is the density function of beta distribution $beta(j+1, n+1-j)$.

Let $y_n(x)$ be a binomial $b(n, x)$ random variable. Then

$$E\left\{y_n(x)\right\} = nt,$$

and

$$var\left\{y_n(x)\right\} = E\left\{y_n(x) - nx\right\}^2 = nx(1-x),$$

and

$$\mathbb{B}_n f(x) = E\left[f\left\{\frac{y_n(x)}{n}\right\}\right],$$

cf. [6].

The classical higher-order Bernoulli polynomials $\mathcal{B}_n^{(v)}(z)$ defined by means of the following generating function

$$F^{(v)}(z,t) = e^{tx}\left(\frac{t}{e^t-1}\right)^v = \sum_{n=0}^{\infty}\mathcal{B}_n^{(v)}(z)\frac{t^n}{n!}. \tag{2.4}$$

The higher-order Bernoulli polynomials play an important role in the finite differences and in (analytic) number theory. So, the coefficients in all the usual cenral-difference formulae for interpolation, numerical differentiation and integration, and differences in terms of derivatives can be expressed in terms of these polynomials cf. ([1], [10], [11], [20]). These polynomials are related to the many branches of Mathematics. By substituting $v = 1$ into the above, we have

$$F(t) = \frac{te^{tx}}{e^t-1} = \sum_{n=1}^{\infty} B_n\frac{t^n}{n!},$$

where $B_n$ is usual Bernoulli polynomials cf. [18].

The usual second kind Stirling numbers with parameters $(n, k)$, denote by $S(n, k)$, that is the number of partitions of the set $\{1, 2, \cdots, n\}$ into $k$ non empty set. For any $t$, it is

well known that the second kind Stirling numbers are defined by means of the generating function cf. ([2], [17], [19])

$$F_S(t, k) = \frac{(-1)^k}{k!}(1 - e^t)^k = \sum_{n=0}^{\infty} S(n, k)\frac{t^n}{n!}. \tag{2.5}$$

These numbers play an important role in many branches of Mathematics, for example, combinatorics, number theory, discrete probability distributions for finding higher order moments. In [7], Joarder and Mahmood demonstrated the application of Stirling numbers of the second kind in calculating moments of some discrete distributions, which are binomial distribution, geometric distribution and negative binomial distribution.

The Hermite polynomials defined by the following generating function:

For $z, t \in \mathbb{C}$,

$$e^{2zt-t^2} = \sum_{n=0}^{\infty} H_n(z)\frac{t^n}{n!}, \tag{2.6}$$

which gives the Cauchy-type integral

$$H_n(z) = \frac{n!}{2\pi i} \int_{\mathcal{C}} e^{2zt-t^2} \frac{dt}{t^{n+1}},$$

where $\mathcal{C}$ is a circle around the origin and the integration is in positive direction cf. [11]. The Hermite polynomials play a crucial role in certain limits of the classical orthogonal polynomials. These polynomials are related to the higher-order Bernoulli polynomials, Gegenbauer polynomials, Laguerre polynomials, the Tricomi-Carlitz polynomials and Buchholz polynomials, cf. [11]. These polynomials also play a crucial role in not only in Mathematics but also in Physics and in the other sciences. In section 4 we give relation between the Hermite polynomials and ($q$-) Bernstein type polynomials.

## 3. Generating Function of the Bernstein type polynomials

Let $\{B_{k,n}(x)\}_{0 \le k \le n}$ be a sequence of Bernstein polynomials. The aim of this section is to construct generating function of the sequence $\{B_{k,n}(x)\}_{0 \le k \le n}$. It is well known that most of generating functions are obtained from the recurrence formulae. However, we do not use the recurrence formula of the Bernstein polynomials for constructing generating function of them.

We now give the following notation:

$$[x] = [x : q] = \begin{cases} \frac{1-q^x}{1-q}, & q \ne 1 \\ x, & q = 1. \end{cases}$$

If $q \in \mathbb{C}$, we assume that $\mid q \mid < 1$.

We define

$$\begin{aligned} F_{k,q}(t, x) &= (-1)^k t^k \exp\left([1 - x]\, t\right) \\ &\times \sum_{m,l=0} \binom{k+l-1}{l} \frac{q^l S(m, k)\, (x \log q)^m}{m!}, \end{aligned} \tag{3.1}$$

where $|q| < 1$, $\exp(x) = e^x$ and $S(m,k)$ denotes the second kind Stirling numbers and

$$\sum_{m,l=0}^{\infty} f(m)g(l) = \sum_{m=0}^{\infty} f(m) \sum_{l=0}^{\infty} g(l).$$

By (3.1), we define the following a new generating function of polynomial $Y_n(k;x;q)$ by

$$F_{k,q}(t,x) = \sum_{n=k}^{\infty} Y_n(k;x;q)\frac{t^n}{n!}, \tag{3.2}$$

where $t \in \mathbb{C}$.

Observe that if $q \to 1$ in (3.2), we have

$$Y_n(k;x;q) \to B_{k,n}(x),$$

hence

$$F_k(t,x) = \sum_{n=k}^{\infty} B_{k,n}(x)\frac{t^n}{n!}.$$

From (3.2), we obtain the following theorem.

**Theorem 1.** *Let $n$ be a positive integer with $k \le n$. Then we have*

$$
\begin{aligned}
Y_n(k;x;q) &= \binom{n}{k}\frac{(-1)^k k!}{(1-q)^{n-k}} \\
&\times \sum_{m,l=0}^{n-k}\sum_{j=0}^{n-k}\binom{k+l-1}{l}\binom{n-k}{k}\frac{(-1)^j q^{l+j(1-x)}S(m,k)\,(x\log q)^m}{m!}.
\end{aligned}
\tag{3.3}
$$

By using (3.1) and (3.2), we obtain

$$
\begin{aligned}
F_{k,q}(t,x) &= \frac{([x]\,t)^k}{k!}\exp([1-x]\,t) \tag{3.4}\\
&= \sum_{n=k}^{\infty} Y_n(k;x;q)\frac{t^n}{n!}.
\end{aligned}
$$

The generating function $F_{k,q}(t,x)$ depends on integer parameter $k$, real variable $x$ and complex variable $q$ and $t$. Therefore the proprieties of this function are closely related to these variables and parameter. By using this function, we give many properties of the ($q$-) Bernstein type polynomials and the other well-known special numbers and polynomials. By applying Mellin transformation to this function, in Section 5, we construct interpolation function of the ($q$-) Bernstein type polynomials.

By the *umbral calculus* convention in (3.4), then we obtain

$$\frac{([x]\,t)^k}{k!}\exp([1-x]\,t) = \exp\left(Y(k;x;q)t\right). \tag{3.5}$$

By using the above, we obtain all recurrence formulae of $Y_n(k;x;q)$ as follows:

$$\frac{([x]\,t)^k}{k!} = \sum_{n=0}^{\infty}\left(Y(k;x;q) - [1-x]\right)^n\frac{t^n}{n!},$$

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where each occurrence of $Y^n(k;x;q)$ by $Y_n(k;x;q)$ (symbolically $Y^n(k;x;q) \to Y_n(k;x;q)$).

By (3.5),

$$[u+v] = [u] + q^u[v]$$

and

$$[-u] = -q^u[u],$$

we obtain the following corollary:

**Corollary 1.** *Let $n$ be a positive integer with $k \le n$. Then we have*

$$Y_{n+k}(k;x;q) = \binom{n+k}{k} \sum_{j=0}^{n}(-1)^j q^{j(1-x)}[x]^{j+k}.$$

**Remark 1.** *By Corollary 1, for all $k$ with $0 \le k \le n$, we see that*

$$Y_{n+k}(k;x;q) = \binom{n+k}{k} \sum_{j=0}^{n}(-1)^j q^{j(1-x)}[x]^{j+k},$$

*or*

$$Y_{n+k}(k;x;q) = \binom{n+k}{k} [x]^k [1-x]^n.$$

*The polynomials $Y_{n+k}(k;x;q)$ are so-called $q$-**Bernstein-type polynomials**. It is easily seen that*

$$\lim_{q \to 1} Y_{n+k}(k;x;q) = B_{k,n+k}(x) = \binom{n+k}{k} x^k (1-x)^n,$$

*which give us (2.2).*

By using derivative operator

$$\frac{d}{dx}\left(\lim_{q \to 1} Y_{n+k}(k;x;q)\right)$$

in (3.1), we obtain

$$\sum_{n=k}^{\infty} \frac{d}{dx}(Y_n(k;x;1)) \frac{t^n}{n!}$$
$$= \sum_{n=k}^{\infty} n Y_{n-1}(k-1;x;1)\frac{t^n}{n!} - \sum_{n=k}^{\infty} n Y_{n-1}(k;x;1)\frac{t^n}{n!}.$$

Consequently, we have

$$\frac{d}{dx}(Y_n(k;x;1)) = nY_{n-1}(k-1;x;1) - nY_{n-1}(k;x;1),$$

or

$$\frac{d}{dx}(B_{k,n}(x)) = nB_{k-1,n-1}(x) - nB_{k,n-1}(x).$$

Observe that by using our generating function, we give different proof of (2.3).

Let $f$ be a function on $[0, 1]$. The ($q$-) Bernstein type polynomial of degree $n$ is defined by

$$\mathbb{Y}_n f(x) = \sum_{j=0}^{n} f\left(\frac{[j]}{[n]}\right) Y_n(j; x; q),$$

where $0 \leq x \leq 1$. $\mathbb{Y}_n$ is called the ($q$-) Bernstein type operator and $Y_n(j; x; q)$, $j = 0, \cdots, n$, defined in (3.3), are called the ($q$-) Bernstein type (basis) polynomials.

## 4. New identities on Bernstein type polynomials, Hermite polynomials and first kind Stirling numbers

**Theorem 2.** *Let $n$ be a positive integer with $k \leq n$. Then we have*

$$Y_n(k; x; q) = [x]^k \sum_{j=0}^{n} \binom{n}{j} \mathcal{B}_j^{(k)}([1-x]) S(n-j, k),$$

*where $\mathcal{B}_j^{(k)}(x)$ and $S(n, k)$ denote the classical higher-order Bernoulli polynomials and the second kind Stirling numbers, respectively.*

*Proof.* By using (2.4), (2.5) and (3.2), we obtain

$$\sum_{n=k}^{\infty} Y_n(k; x; q)\frac{t^n}{n!}$$

$$= [x]^k \sum_{n=0}^{\infty} S(n, k)\frac{t^n}{n!} \sum_{n=0}^{\infty} \mathcal{B}_j^{(k)}([1-x])\frac{t^n}{n!}.$$

By using Cauchy product in the above, we have

$$\sum_{n=k}^{\infty} Y(k, n; x; q)\frac{t^n}{n!}$$

$$= [x]^k \sum_{n=0}^{\infty} \sum_{j=0}^{n} \mathcal{B}_j^{(k)}([1-x]) S(n-j, k)\frac{t^n}{j!\,(n-j)!}.$$

From the above, we have

$$\sum_{n=k}^{\infty} Y_n(k; x; q)\frac{t^n}{n!} \tag{4.1}$$

$$= [x]^k \sum_{n=0}^{k-1} \sum_{j=0}^{n} \mathcal{B}_j^{(k)}([1-x]) S(n-j, k)\frac{t^n}{j!\,(n-j)!}$$

$$+ [x]^k \sum_{n=k}^{\infty} \sum_{j=0}^{n} \mathcal{B}_j^{(k)}([1-x]) S(n-j, k)\frac{t^n}{j!\,(n-j)!}.$$

By comparing coefficients of $t^n$ in the both sides of the above equation, we arrive at the desired result. $\square$

**Remark 2.** *In* [5], *Gould gave a different relation between the Bernstein polynomials, generalized Bernoulli polynomials and the second kind Stirling numbers. Oruc and Tuncer* [13] *gave relation between the q-Bernstein polynomials and the second kind q-Stirling numbers. In* [12], *Nowak studied on approximation properties for generalized q-Bernstein polynomials and also obtained Stancu operators or Phillips polynomials.*

From (4.1), we get the following corollary:

**Corollary 2.** *Let $n$ be a positive integer with $k \leq n$. Then we have*

$$[x]^k \sum_{n=0}^{k-1} \sum_{j=0}^{n} \frac{\mathcal{B}_j^{(k)}\left([1-x]\right) S(n-j,k)}{j!\,(n-j)!} = 0.$$

**Theorem 3.** *Let $n$ be a positive integer with $k \leq n$. Then we have*

$$H_n(1-y) = \frac{k!}{y^k} \sum_{n=0}^{\infty} Y_{n+k}(k;y;q) \frac{2^n}{(n+k)!}.$$

*Proof.* By (2.6), we have

$$e^{2zt} = \sum_{n=0}^{\infty} \frac{t^{2n}}{n!} \sum_{n=0}^{\infty} H_n(z) \frac{t^n}{n!}.$$

By Cauchy product in the above, we obtain

$$e^{2zt} = \sum_{n=0}^{\infty} \left( \sum_{j=0}^{n} \binom{n}{j} H_j(z) \right) \frac{t^{2n-j}}{n!}. \tag{4.2}$$

By substituting $z = 1-y$ into (4.2), we have

$$\sum_{n=0}^{\infty} \left( \sum_{j=0}^{n} \binom{n}{j} H_j(1-y) \right) \frac{t^{2n-j}}{n!} = \frac{k!}{y^k} \sum_{n=0}^{\infty} \left(2^n Y_{n+k}(k;y;q)\right) \frac{t^n}{(n+k)!}.$$

By comparing coefficients of $t^n$ in the both sides of the above equation, we arrive at the desired result. $\square$

## 5. INTERPOLATION FUNCTION OF THE $(q\text{-})$ BERNSTEIN TYPE POLYNOMIALS

The classical Bernoulli numbers interpolate by Riemann zeta function, which has a profound effect on number theory and complex analysis. Thus, we construct interpolation function of the $(q\text{-})$ Bernstein type polynomials.

For $z \in \mathbb{C}$, and $x \neq 1$, by applying the Mellin transformation to (3.1), we get

$$S_q(z,k;x) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{z-k-1} F_{k,q}(-t,x)dt.$$

By using the above equation, we defined interpolation function of the polynomials, $Y_n(k;x;q)$ as follows:

**Definition 1.** *Let $z \in \mathbb{C}$ and $x \neq 1$. We define*

$$S_q(z, k; x) = (1 - q)^{z-k} \sum_{m,l=0} \binom{z + l - 1}{l} \frac{q^{l(1-x)} S(m, k) (x \log q)^m}{m!}. \qquad (5.1)$$

By using (5.1), we obtain

$$S_q(z, k; x) = \frac{(-1)^k}{k!} [x]^k [1 - x]^{-z},$$

where $z \in \mathbb{C}$ and $x \neq 1$.

By (5.1), we have $S_q(z, k; x) \to S(z, k; x)$ as $q \to 1$. Thus we have

$$S(z, k; x) = \frac{(-1)^k}{k!} x^k (1 - x)^{-z}.$$

By substituting $x = 1$ into the above, we have

$$S(z, k; 1) = \infty.$$

We now evaluate the $m$th $z$-derivatives of $S(z, k; x)$ as follows:

$$\frac{\partial^m}{\partial z^m} S(z, k; x) = \log^m \left( \frac{1}{1 - x} \right) S(z, k; x), \qquad (5.2)$$

where $x \neq 1$.

By substituting $z = -n$ into (5.1), we obtain

$$S_q(-n, k; x) = \frac{1}{(1 - q)^n} \sum_{m,l=0} \binom{-n + l - 1}{l} \frac{q^{l(1-x)} S(m, k) (x \log q)^m}{m!}.$$

By substituting (3.3) into the above, we arrive at the following theorem, which relates the polynomials $Y_{n+k}(k; x; q)$ and the function $S_q(z, k; x)$.

**Theorem 4.** *Let $n$ be a positive integer with $k \leq n$ and $0 < x < 1$. Then we have*

$$S_q(-n, k; x) = \frac{(-1)^k n!}{(n + k)!} Y_{n+k}(k; x; q).$$

**Remark 3.**

$$\begin{aligned}
\lim_{q \to 1} S_q(-n, k; x) &= S(-n, k; x) \\
&= \frac{(-1)^k n!}{(n + k)!} x^k (1 - x)^n \\
&= \frac{(-1)^k n!}{(n + k)!} B_{k, n+k}(x).
\end{aligned}$$

*Therefore, for $0 < x < 1$, the function*

$$S(z, k; x) = \frac{(-1)^k}{k!} x^k (1 - x)^{-z}$$

*interpolates the classical Bernstein polynomials of degree $n$ at negative integers.*

By substituting $z = -n$ into (5.2), we obtain the following corollary.

**Corollary 3.** *Let $n$ be a positive integer with $k \leq n$ and $0 < x < 1$. Then we have*

$$\frac{\partial^m}{\partial z^m} S(-n, k; x) = \frac{(-1)^k n!}{(n+k)!} B_{k,n+k}(x) \log^m \left( \frac{1}{1-x} \right).$$

## 6. FURTHER REMARKS AND OBSERVATION

The Bernstein polynomials are used for important applications in many branches of Mathematics and the other sciences, for instance, approximation theory, probability theory, statistic theory, number theory, the solution of the differential equations, numerical analysis, constructing Bezier curve, $q$-analysis, operator theory and applications in computer graphics. Thus we look for the applications of our new functions and the ($q$-) Bernstein type polynomials. Due to Oruc and Tuncer [13], the $q$-Bernstein polynomials shares the well-known shape-preserving properties of the classical Bernstein polynomials. When the function $f$ is convex then

$$\beta_{n-1}(f, x) \geq \beta_n(f, x) \text{ for } n > 1 \text{ and } 0 < q \leq 1,$$

where

$$\beta_n(f, x) = \sum_{r=0}^{n} f_r \begin{bmatrix} n \\ r \end{bmatrix} x^r \prod_{s=0}^{n-r-1} (1 - q^s x)$$

and

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{[n] \cdots [n-r+1]}{[r]!}.$$

As a consequence of this one can show that the approximation to convex function by the $q$-Bernstein polynomials is one sided, with $\beta_n f \geq f$ for all $n$. $\beta_n f$ behaves is very nice way when one vary the parameter $q$. In [1], the authors gave some applications on the approximation theory related to Bernoulli and Euler polynomials.

We conclude this section by the following questions:

1) *How can one demonstrate approximation by ($q$-) Bernstein type polynomials, $Y_{n+k}(k; x; q)$?*

2) *Is it possible to define uniform expansions of the ($q$-) Bernstein type polynomials, $Y_{n+k}(k; x; q)$?*

3) *Is it possible to give applications of the ($q$-) Bernstein type polynomials in calculating moments of some distributions in Statistics, $Y_{n+k}(k; x; q)$?*

4) *How can one give relations between the ($q$-) Bernstein type polynomials, $Y_{n+k}(k; x; q)$ and the Milnor algebras.*

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